THE PROPAGATION OF WAVES IN INFINITE PLATES

(O RASPROSTRANENII VOLN V BESKONECHNYKH PLITAKH)

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In his paper [1] Ufliand considers, among other problems, the effect of concentrated forces on an infinite plate. He solves the equation by taking into account the rotatory inertia of the plate elements and the influence of the shear forces, but the boundary conditions were formulated in agreement with the elementary theory. The present paper studies the effect of annular pressure on an infinite plate with consideration of rotatory inertia and shear deformation. A comparison is made between the shear forces, calculated on the basis of the elementary theory, for the problem posed in the paper [1] and of the problem formulated below.

The system of equations, deduced by Ufliand in paper [1] in polar coordinates for the case when the unknown functions do not depend on the polar angle and there is no external loading, in terms of non-dimensional quantities assume the form

$$\Delta w - \frac{\partial \alpha_r}{\partial r} - \frac{\alpha_r}{r} = \gamma \frac{\partial^2 w}{\partial t^2} \qquad \left(\gamma = \frac{q}{k(1-\nu)}\right) \tag{1}$$

$$\Delta \alpha_r - \frac{\alpha_r}{r^2} + \frac{12}{\gamma}\left(\frac{\partial w}{\partial r} - \alpha_r\right) = \frac{\partial^2 \alpha_r}{\partial t^2}$$

Here the linear dimensions refer to h, the mass to ρh^3, the time to h/v_1, forces to E, bending moments to Eh^2, stresses to E, the velocity to v_1. The notation is as follows: w is the deflection of the central plane of the plate, ρ - the mass density, h - the thickness of the plate, E - the modulus of elasticity, k - the coefficient depending on the shape of the cross-section, α - the angle of rotation of a plate element as a body in the rz-plane, where r is the non-dimensional coordinate, v_1 and v_2 the velocities of propagation of waves in the plate

$$v_1 = \sqrt{\frac{E}{\rho(1-\nu^2)}}, \qquad v_2 = \sqrt{\frac{k\mu}{\rho}}.$$

The system (1) will be solved by operational methods for zero initial conditions by introducing

$$W = \int_0^\infty w e^{-pt}\,dt, \qquad A_r = \int_0^\infty \alpha_r e^{-pt}\,dt \tag{2}$$

In the transformed plane the solutions of the system will be

$$\begin{gathered} W = C_1 K_0(n_1 r) + C_2 K_0(n_2 r) + C_3 I_0(n_1 r) + C_4 I_0(n_2 r) \\ A_r = \frac{n_1^2 - p^2\gamma}{n_1}\left[C_3 I_1(n_1 r) - C_1 K_1(n_1 r)\right] + \frac{n_2^2 - p^2\gamma}{n_2}\left[C_4 I_1(n_2 r) - C_2 K_1(n_2 r)\right] \end{gathered} \tag{3}$$

where

$$n_{1,2}^2 = \frac{\gamma+1}{2}p^2 \mp \frac{\gamma-1}{2}p\sqrt{p^2 - a^2}, \qquad a^2 = \frac{48}{(\gamma-1)^2}$$

K_0, K_1 are Macdonald functions and I_0, I_1 are Bessel functions for imaginary arguments.

Let there be cut into the infinite plate a circular hole of small radius r_0 with the center at the origin of coordinates. It will be assumed that a force Q is distributed over the circumference of the circle of radius r_0. The boundary conditions will be written in the form

$$r_0\left(\frac{\partial W}{\partial r} - \alpha_r\right) = \frac{1+\nu}{k\pi}Q, \qquad \alpha_r = 0 \qquad \text{for } r = r_0 \tag{4}$$

Since the solution of the system (1) must vanish at inifinity, one must have $C_3 = C_4 = 0$, and the constants C_1 and C_2 follow from the conditions for the transformed solutions (4)

$$\begin{gathered} C_1 = \frac{(1+\nu)}{2\pi k\gamma}\frac{Q}{2}\left(\frac{1}{p^2\sqrt{p^2-a^2}} - \frac{1}{p^3}\right)\frac{n_1}{K_1(n_1 r_0)} \\ C_2 = -\frac{(1+\nu)}{2\pi k\gamma}\frac{Q}{r_0}\left(\frac{1}{p^2\sqrt{p^2-a^2}} + \frac{1}{p^3}\right)\frac{n_2}{K_1(n_2 r_0)} \end{gathered} \tag{5}$$

The transform for the quantity $2(1+\nu)N_r/k$ will be

$$\frac{dW}{dr} - A_r = \frac{1+\nu}{2\pi k r_0}Q\left[\left(-\frac{1}{\sqrt{p^2-a^2}} + \frac{1}{p}\right)\frac{K_1(n_1 r)}{K_1(n_1 r_0)} + \left(\frac{1}{\sqrt{p^2-a^2}} + \frac{1}{p}\right)\frac{K_1(n_2 r)}{K_1(n_2 r_0)}\right] \tag{6}$$

In order to find the shear forces, the inversion formula will be applied. It is necessary to evaluate the integrals

$$I_1 = \frac{1}{2\pi i}\int_L \frac{K_1(n_\nu r)\, e^{aqt}\,dq}{\sqrt{q^2-1}\,K_1(n_\nu r_0)}, \qquad I_2^{(\nu)} = \frac{1}{2\pi i}\int \frac{K_1(n_\nu r)\, e^{aqt}\,dq}{qK_1(n_\nu r_0)} \qquad \left(q = \frac{p}{a}\right) \tag{7}$$

where L is the Riemann-Mellin contour. In paper [1] the integrals have been evaluated in a similar manner. Their calculation is required for $t > (r - r_0)/v_\nu$, since for $t < (r - r_0)/v_\nu$ they vanish by Jordan's Lemma. The integrands in (7) have the same branch points as in reference [1]; in fact, The evaluation of the integrals (7) may be executed along the same contours as in reference [1] by use of Table 1, given there. One only has to keep in mind that the second integral in (7) does not vanish for a small circle of radius ϵ with center at $q = 0$ as $\epsilon \to 0$. One obtains

$$I_1^{(2)} = -\frac{1}{\pi}\int_0^1 \left\{ e^{a\rho t}\,\mathrm{Re}\,\frac{K_1[P\sqrt{\rho}(\lambda + i\mu)r]}{K_1[P\sqrt{\rho}(\lambda + i\mu)r_0]} + \right.$$

$$\left. + e^{-a\rho t}\,\mathrm{Re}\,\frac{K_1[P\sqrt{\rho}(-\lambda + i\mu)r]}{K_1[P\sqrt{\rho}(-\lambda + i\mu)r_0]}\right\}\frac{d\rho}{\sqrt{1-\rho^2}} \tag{8}$$

$$I_2^{(2)} = \frac{1}{\pi}\int_0^1 \left\{ e^{a\rho t}\,\mathrm{Im}\,\frac{K_1[P\sqrt{\rho}(\lambda + i\mu)r]}{K_1[P\sqrt{\rho}(\lambda + i\mu)r_0]} - e^{-a\rho t}\,\mathrm{Im}\,\frac{K_1[P\sqrt{(\rho}(-\lambda + i\mu)r]\,d\rho}{K_1[P\sqrt{\rho}(-\lambda + i\mu)r_0]\,\rho} + \frac{r_0}{r}\right.$$

$$I_1^{(1)} = I_1^{(2)} - I_1^*, \qquad I_2^{(1)} = -I_2^{(2)} + I_2^* + \frac{2r_0}{2} \tag{9}$$

Here

$$I_1^* = \int_0^\alpha \Phi\,[\pi I_1(P\omega\sqrt{\rho}\,r_0) - K_1(P\omega\sqrt{\rho}\,r_0)]\,\frac{d\rho}{\sqrt{\rho^2+1}}$$

$$I_2^* = -\int_0^\alpha \Phi[\pi_1 I_1(P\omega\sqrt{\rho}r_0) - K_1(P\omega\sqrt{\rho}r_0)]\,\frac{d\rho}{\rho} \tag{10}$$

$$\Phi = \frac{K_1(P\omega\sqrt{\rho}\,r)\,I_1(P\omega\sqrt{\rho}\,r_0) - I_1(P\omega\sqrt{\rho}\,r)\,K_1(P\omega\sqrt{\rho}\,r_0)}{K_1(P\omega\sqrt{\rho}\,r_0)[K_1^2(P\omega\sqrt{\rho}\,r_0) + \pi^2 I_1^2(P\omega\sqrt{\rho}\,r_0)]}\,.$$

Here and below the following notation has been introduced

$$q = \rho e^{i\varphi}, \quad P = aM, \quad M^2 = \tfrac{1}{2}(\gamma+1), \quad N = \frac{\gamma-1}{\gamma+1}, \quad v_1 = 1, \quad v_2 = \frac{1}{\sqrt{\gamma}}$$

$$\omega = \sqrt{N\sqrt{\rho^2-1}-\rho}, \quad \lambda = \sqrt{\frac{s+\rho}{2}}, \quad \mu = \sqrt{\frac{s-\rho}{2}}, \quad s = \sqrt{1-N^2}\sqrt{\rho^2+\alpha^2}$$

Then the expressions for the shearing forces for the different regions will be

$$N_r = \frac{Q}{4\pi r_0}\left(I_1^* + I_2^* + \frac{2r_0}{r}\right) \qquad \text{for } r - r_0 < \frac{t}{\sqrt{\gamma}}$$

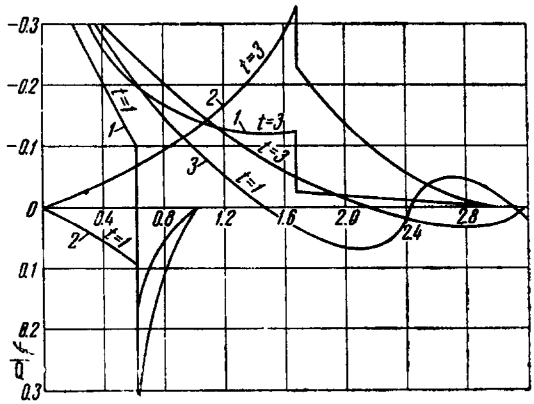
$$N_r = \frac{Q}{4\pi r_0}\left(-I_1^{(2)} + I_1^* - I_2^{(2)} + I_2^* + \frac{2r_0}{r}\right) \qquad \text{for } \frac{t}{\sqrt{\gamma}} < r - r_0 < t \tag{11}$$

$$N_r = 0 \qquad \text{for } r - r_0 > t$$

By passing, in the formulas (11), to the limit as $r_0 \to 0$, one may obtain formulas for a concentrated force acting at the center of the plate.

For $(r - r_0) < t/\gamma$ 1/2, for example, the formula will be:

The figure shows the graphs of the shear forces for the two instants $t = 1$ and $t = 3$ by the two theories. In this figure, 1 refers to the wave



theory for the exact boundary conditions, 2 to the wave theory for the boundary conditions of the elementary theory, 3 to the elementary theory. The qualitative difference of the wave theories from the elementary theories here consists in the fact that as in the last theory N_r changes continuously from ∞ to 0 as r goes from 0 to ∞, in the wave solutions N_r undergoes discontinuities at the front of the wave v_2 of magnitude $Q/2r$ and $N_r = 0$ in front of the wave v_1. Further, it is seen from the formulation of the problem that as $r \to 0$, $N_r \to \infty$ in 3 and 1 and $N_r \to 0$ in 2. For large r, the values of N_r, evaluated from (12) approach the value calculated in [1], when $t = 3$, but for $t = 1$ they differ significantly.

The property of N_r to tend to infinity as $r \to 0$ contradicts the physical nature of the deformations of the plate. This originates from the fact that initially the element, in accordance with the accepted theory, undergoes a pure shear deformation. The contradiction may be removed, if one considers the propagation of the deformation of a cylinder, singled out in the neighborhood of the acting forces, as has been done by Timoshenko [2].

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